Kuwait University | Math 101 | 22 December 2010
Department of Mathematics | Second Exam | 6.30–8pm

Answer all questions. Calculators and mobile phones are NOT allowed.

1. (3pts) Suppose a curve is given by $x^2y + 5 = -\frac{1}{2}xy^3$. Find the slope of the tangent line to the curve at the point $(2, -1)$.

2. (4pts) A point $P(x, y)$ moves on the line $y = 2x - 1$ so that $\frac{dx}{dt} =$ 5cm/sec. Find the rate at which the distance between P and $Q(-1, 1)$ is changing when P is at $(0, -1)$.

3. (3pts) Use differentials to approximate $4 - (8.1)^{1/3}$.

4. (4pts) Suppose $f(x) = x - 2\sin x$. Find the absolute maximum and minimum values of f on the interval $[0, \pi]$.

5. (3pts)
   (a) State the Mean value Theorem
   (b) Let f be continuous on $[a, b]$ with $f'(x) < 0$ for all $x \in (a, b)$. Show that $f(b) < f(a)$.

6. (8pts) Let $f(x) = \frac{x-1}{x^3}$. You are given that $f' = \frac{3-2x}{x^4}$ and $f'' = \frac{6x-12}{x^5}$.
   (a) Find the horizontal and verticals asymptotes, if any.
   (b) Find the intervals on which f is increasing and the intervals on which f is decreasing. Find the local extrema of f, if any.
   (c) Find the intervals on which f is concave upwards and the intervals on which f is concave downwards. Find the inflection points of f, if any.
   (d) Sketch the graph of f.

Solutions

1. (3pt) We have $2xy + x^2y' = -\frac{1}{2}(y^3 + 3xy^2y')$. When $x = 2, y = -1$ we have $y' = \frac{9}{14}$.

2. (4pts) Since $y = 2x - 1$, we have $y' = 2x' = 10$. Let L be the distance between P and Q. Then $L^2 = (x+1)^2 + (y-1)^2$, so that $LL' = (x+1)x' + (y-1)y' = 5(x+1) + 10(y-1)$. When $x = 0, y = -1$ we have $L = \sqrt{5}$ and so $L' = -15/\sqrt{5}$.

3. (3pts) Put $y = f(x) = 4 - x^{1/3}$. Then $dy = -\frac{1}{3x^{2/3}}dx$. When $x = 8, dx = 0.1$ and $dy = -\frac{1}{120}$. Thus $4 - (8.1)^{1/3} \approx 2 + dy = 2 - \frac{1}{120}$.

4. (4pts) $f' = 1 - 2\cos x = 0 \iff \cos x = 1/2 \iff x = \pi/3$. Now $f(0) = 0, f(\pi/3) = \pi/3 - \sqrt{3}, f(\pi) = \pi$. Thus max is π, min $\pi/3 - \sqrt{3}$.

5. (b) By MVT, there is $c \in (a, b)$ with $f(b) - f(a) = f'(c)(b - a) < 0$ since $f' < 0$ and $b - a > 0$. Thus $f(b) < f(a)$.

6. (a) $\lim_{x\to\pm\infty} y = 0$, so $y = 0$ is horizontal asymptote and $\lim_{x\to 0^-} y = \infty$, so $x = 0$ is vertical asymptote.

   (b) $y' = \frac{3-2x}{x^4} > 0$ if $x < 3/2, 0$ if $x = 3/2$ and < 0 if $x > 3/2$. The only critical point $(\frac{3}{2}, \frac{4}{27})$ is a local max and f is increasing for $x < 3/2$, decreasing for $x > 3/2$.

   (c) $y'' = \frac{6x-12}{x^5} > 0$ for $x > 2$ or $x < 0$ and $y'' < 0$ on $(0, 2)$. Thus $(2, \frac{1}{8})$ is inflection point. Curve is concave up for $x < 0$ or $x > 2$ and down for $x \in (0, 2)$.

   (d)